Pole Diagrams

1. Definition of the Pole Diagram

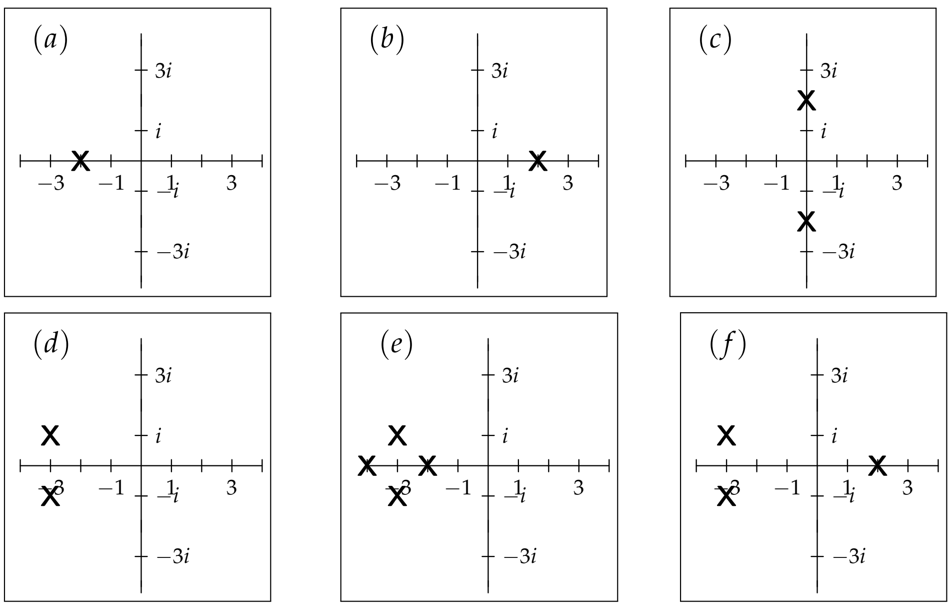
The **pole diagram** of a function $F(s)$ is simply the complex s-plane with an **X** marking the location of each pole of $F(s)$.

Example 1. Draw the pole diagrams for each of the following functions.

a) $F_1(s) = \frac{1}{s+2}$ b) $F_2(s) = \frac{1}{s-2}$ c) $F_3(s) = \frac{1}{s^2+4}$

d) $F_4(s) = \frac{s}{s^2+6s+10}$ e) $F_5(s) = \frac{1}{((s^2+3)^2+1)(s+2)(s+4)}$ f) $F_6(s) = \frac{1}{((s+3)^2+1)(s-2)}$

Solution.



For (d) we found the poles by first completing the square: $s^2 + 6s + 10 = (s+3)^2 + 1$, so the poles are at $s = -3 \pm i$.

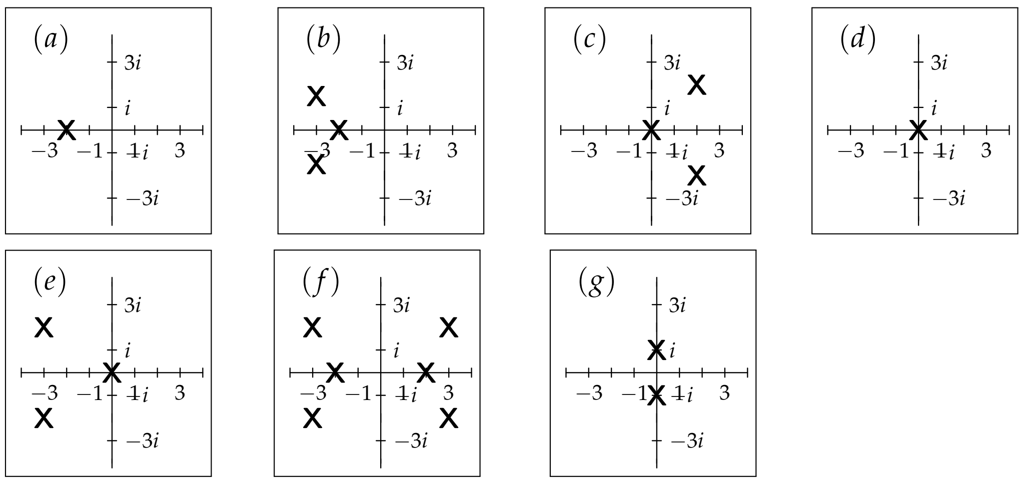
Example 2. Use the pole diagram to determine the exponential growth rate of the inverse Laplace transform of each of the functions in example 1.

Solution.

a) The largest pole is at -2, so the exponential growth rate is -2.
b) The largest pole is at 2, so the exponential growth rate is 2.
c) The poles are $\pm 2i$, so the largest real part of a pole is 0. The exponential growth rate is 0.
d) The largest real part of a pole is -3. The exponential growth rate is -3.
e) The largest real part of a pole is -2. The exponential growth rate is -2.

f) The largest real part of a pole is 2. The exponential growth rate is 2.

Example 3. Each of the pole diagrams below is for a function $F(s)$ which is the Laplace transform of a function $f(t)$. Say whether
(i) $f(t) \to 0$ as $t \to \infty$
(ii) $f(t) \to \infty$ as $t \to \infty$
(iii) You don't know the behavior of $f(t)$ as $t \to 0$,



Solution. a) Exponential growth rate is -2, so $f(t) \to 0$.
b) Exponential growth rate is -2, so $f(t) \to 0$.
c) Exponential growth rate is 2, so $f(t) \to \infty$.
d) Exponential growth rate is 0, so we can't tell how $f(t)$ behaves.
Two examples of this: (i) if $F(s) = 1/s$ then $f(t) = 1$, which stays bounded; (ii) if $F(s) = 1/s^2$ then $f(t) = t$, which does go to infinity, but more slowly than any positive exponential.
e) Exponential growth rate is 0, so don't know the behavior of $f(t)$.
f) Exponential growth rate is 3, so $f(t) \to \infty$.
g) Exponential growth rate is 0, so don't know the behavior of $f(t)$. (e.g. both $\cos t$ and $t \cos t$ have poles at $\pm i$.

2. The Pole Diagram for an LTI System

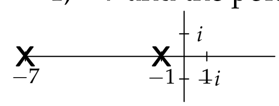
Definition: The pole diagram for an LTI system is defined to be the pole diagram of its transfer function.

Example 4. Give the pole diagram for the system

$$\ddot{x} + 8\dot{x} + 7x = f(t),$$

where we take $f(t)$ to be the input and $x(t)$ the output.

Solution. The transfer function for this system is $W(s) = \dfrac{1}{s^2+8s+1} = \dfrac{1}{(s+1)(s+7)}$. Therefore, the poles are $s = -1, -7$ and the pole diagram is



Example 5. Give the pole diagram for the system

$$\ddot{x} + 4\dot{x} + 6x = \dot{y},$$

where we consider $y(t)$ to be the input and $x(t)$ to be the output.

Solution. Assuming rest IC's, Laplace transforming this equation gives us $(s^2+4s+6)X = sY$. This implies $X(s) = \dfrac{s}{s^2+4s+6}Y(s)$ and the transfer function is $W(s) = \dfrac{s}{s^2+4s+6}$. This has poles at $s = -2 \pm \sqrt{2}\,i$.

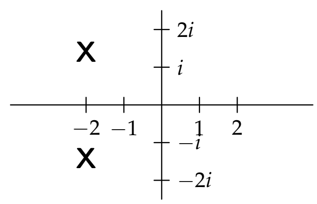


Figure: Pole diagram for the system in example 5.